Exercise 1.9

Saturday, 9 September 2023 19:55

The proof is very similar (and partially taken from / based on) to the proof of theorem 1.7.

"We consider a machine M_{i^*} that has the maximum load, and we consider the last job j^* scheduled on M_{i^*}. If $j^* \leq m$, then j^* is the only job scheduled on M_{i^*} - this is true because the greedy algorithm schedules the first m jobs on different machines. Hence, our algorithm is optimal in this case. Now consider the case $j^* > m$."

Here, we can derive that

the load before assigning job j^* is equal to the load of jobs 1 to j^*-1

$$\mathrm{load}'(M_{i^*}) \leq \frac{1}{m}\sum_{1\leq i\leq m} \mathrm{load}'(M_i) = \frac{1}{m}\sum_{1\leq j<j^*} t_j \leq \frac{1}{m}\left(\left(\sum_{1\leq j\leq n} t_j\right) - t_{j^*}\right)$$

the load without job j^*

the load on M_{i^*} before assigning j^* must be below average

that load is at most the load of all jobs, minus that of j^*.

Thus, we get that

$$\mathrm{load}(M_{i^*}) = \mathrm{load}'(M_{i^*}) + t_{j^*} \leq \frac{1}{m}\left(\sum_{1\leq j\leq n} t_j\right) + \left(1-\frac{1}{m}\right)t_{j^*}$$

From lemma 1.3, it follows that the first term is bounded by OPT; thus, we get

$$\mathrm{load}(M_{i^*}) \leq OPT + \left(1-\frac{1}{m}\right)t_{j^*}$$

From lemma 1.6, we have that $t_m + t_{m+1} \leq OPT$. Thus, we also have that since $j^* > m$, and the jobs are ordered:

$$t_{j^*} \overset{\text{since } t_j\leq t_i \text{ for } i\leq j, \text{ and } j^*\geq m+1}{\leq} t_{m+1} \overset{\text{since } t_{m+1}\leq t_m}{\leq} \frac{t_m+t_{m+1}}{2} \leq \frac{OPT}{2}$$

Combining the previous two inequalities, we get

$$\begin{aligned}\mathrm{load}(M_{i^*}) &\leq OPT + \left(1-\frac{1}{m}\right)\frac{1}{2}OPT\\ &= OPT\left(1+\frac{1}{2}-\frac{1}{2m}\right)\\ &= \left(\frac{3}{2}-\frac{1}{2m}\right)OPT\end{aligned}$$

This proves the total load on M_{i^*} is at most $\left(\frac{3}{2}-\frac{1}{2m}\right)OPT$. Hence, Ordered-Scheduling must be a $\left(\frac{3}{2}-\frac{1}{2m}\right)$-approximation algorithm. □