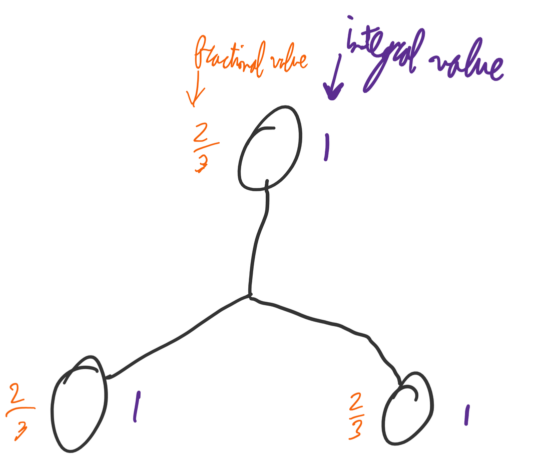
Exercise 3.5 (i)-(iii)

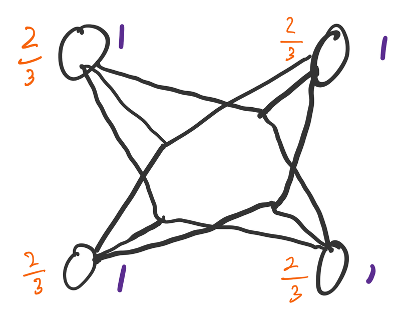
Sunday, 10 September 2023 19:36

1. Rough idea: same as 0/1-LP for vertex-cover, with following changes:
   a. Instead of the constraint $x_u + x_v \geq 1$, where $(u, v) \in E$ in the regular vertex-cover problem, we add the constraint $\sum_{v \in e} x_v \geq 2$ for each d-edge e. This states that, for every d-edge, at least two vertices adjacent to it should be added.
2. As threshold in the rounding scheme, we take $\frac{1}{d-1}$. This will always give a valid solution, since, in the 'worst' case (i.e. one in which the sum of the two highest variables (a.k.a. the vertices which cover the edge) is minimal), one variable takes value 1, and all the others take value $\frac{1}{d-1}$. In other words, there are at least two vertices in each edge with decision value variable $\frac{1}{d-1}$.
   a. Approximation ratio can be obtained by taking the solution to the relaxed LP as lower bound, then filling in the inequalities in a similar way to the proof of theorem 3.4.
      i. Note: the factor 2 appearing in that proof probably needs to be replaced by $\frac{1}{\frac{1}{d-1}} = d - 1$. This suggests that the approximation ratio could quite well be $d - 1$. ~~However, this proof is not complete, since this cannot be true for $d = 2$; otherwise, we would have an optimal algorithm for the regular weighted vertex-cover problem.~~ Never mind: we are talking about double vertex cover here, and that problem always has the property that the optimal solution for $d = 2$ consists of selecting all vertices which are part of an edge.
3. Probably, this can be found by considering that the decision variables for the two selected vertices (in the 'best' case, only two vertices are selected per edge) need to have a value of at least $\frac{2}{d}$, i.e. the average value of all decision variables for a given edge, given that their total combined value is at least 2.
   Furthermore, in the non-relaxed (i.e. integer) LP, we have that the total value of a solution (for a given edge) must be at least 2.
   Note: I probably need to find an example which leads to this integrality gap. See left for some examples which suggest the integrality gap for $d = 3$ is at least $\frac{2 \cdot 1}{2 \cdot \frac{2}{3}} = \frac{2}{\left(\frac{4}{3}\right)} = \frac{6}{4} = \frac{3}{2}$.



$$IG = \frac{2}{\left(\frac{4}{3}\right)} = \frac{6}{4} = \frac{3}{2}$$

note: we need 2 nodes to obtain a valid solution



(i) & (ii)

double-vertex-cover(V, E)

1. $d \leftarrow$ number of elements in the first element of E. ← it doesn't matter which one, but the first element is guaranteed to exist
2. $n \leftarrow |V|$ number of vertices
3. Solve the relaxed linear program corresponding to the given problem:

   Minimize $\sum_{i=1}^{n} x_i$

   Subject to $\sum_{i \in \{1 \leq i \leq n \mid v_i \in e_j\}} x_i \geq 2$ for all edges $e_j \in E$ This notation assumes vertices are numbered as $v_1, \dots, v_n$

   $0 \leq x_i \leq 1$ for all $1 \leq i \leq n$

   rough idea: make sure that, on each edge, the decision variables for the vertices on that edge sum to two. We then minimize the value of the sum of the decision variables.
4. solution $\leftarrow \{v_i \in V \mid x_i \geq \frac{1}{d-1}\}$
5. return solution

Proof of correctness For this algorithm to be correct, we must have that, for every d-edge, at least two vertices are included in the cover. Since the rounding threshold is constant, we know that, for two vertices to be included in the cover, the two vertices with the highest decision variable values must both be at least $\frac{1}{d-1}$.
First of all, we note that, for the first constraint to be satisfied, we must have that all decision variables for a given edge have a value of at least $\frac{2}{d}$, as spreading the minimum total value of 2 over all d variables leads to the lowest maximal value among them. However, since $\frac{2}{d} \geq \frac{1}{d-1}$ for $d \geq 2$, we have that the at least one variable must have a value above the threshold. To see why a second variable must be above the threshold as well, we note that, by the second constraint (i.e. $0 \leq x_i \leq 1$ for all $1 \leq i \leq n$), the value of the highest-valued decision variable can be at most 1. Then, for the first constraint to be satisfied, we need that the average value of the $d-1$ decision variables with the lowest values is at least $\frac{\boxed{2-1}}{d-1} = \frac{1}{d-1}$. → the value remaining for the lower variables after subtracting the maximal value for the highest one
From this, it follows that at least one of the $d-1$ variables with the lowest values has a value above or equal to the threshold. This implies at least two vertices for each d-edge are included in the cover, which proves the correctness of said cover.

Approximation ratio We have that $OPT \geq OPT_{relaxed} = \sum_{i=1}^{n} x_i$

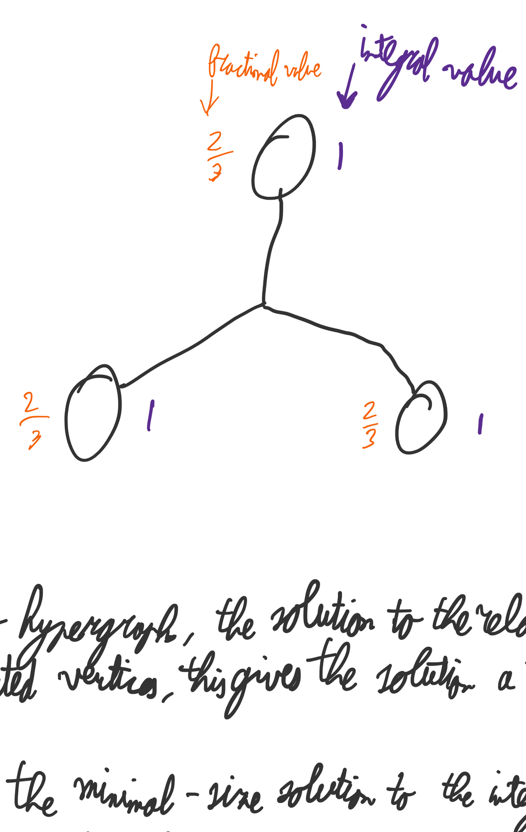
Furthermore, we have that the total size of the solution to the relaxed LP, after rounding, can be bounded as follows: we have that the decision variables for each of these vertices must have value at least $\frac{1}{d-1}$ (as explained in the correctness proof above). From this, it follows that

note: since $x_i \geq \frac{1}{d-1}$ for $v_i \in$ solution, $x_i \cdot (d-1) \geq 1$

$$\begin{aligned} \text{double-vertex-cover}(V, E) = \sum_{v_i \in \text{solution}} 1 &\leq \sum_{v_i \in \text{solution}} x_i \cdot (d-1) \\ &\leq (d-1) \sum_{v_i \in \text{solution}} x_i \\ &\leq (d-1) \sum_{i=1}^{n} x_i \\ &\leq (d-1)\, OPT_{relaxed} \\ &\leq (d-1)\, OPT \end{aligned}$$

This proves double-vertex-cover is a $(d-1)$-approximation algorithm

(iii) Consider the example below:



On this 3-hypergraph, the solution to the relaxed problem will assign value $\frac{2}{3}$ to all vertices. On the selected vertices, this gives the solution a total size of $\frac{4}{3}$, since 2 vertices need to be selected.

However, the minimal-size solution to the integer LP has size 2, when 2 vertices are assigned value 1 (and are then selected).

This gives that the integrality gap must be at least $\frac{2}{\left(\frac{4}{3}\right)} = \frac{6}{4} = \frac{3}{2}$.

But, we can do better. Take a case with n vertices, and the edges

$\{v_1, v_i, v_j\}$, for $1 \leq i < j \leq n$