Exercise 4.1

donderdag 14 september 2023 15:43

Probably not: a counterexample probably consists of having values round down to 0, which lets them be ignored by the algorithm.

As an example, take the following items:

	Value	Weight
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	0.9	0.2
	1.999...	5

Also, take $W = 5$. Then, for $\Delta = 1$, which can be obtained using
$1 = \Delta = \frac{\epsilon}{n} \cdot \mathrm{LB}$
$\epsilon = \frac{n\Delta}{LB} = \frac{n\Delta}{\max_{1\le i\le n} value(x_i)} = \frac{7\cdot 1}{1.999...} = \frac{7}{1.99...} \approx 3.5$, the algorithm will clearly return the item with a value of 1.9 as the highest-value subset of items *(since all other items have their value rounded down to 0, they are ignored by the algorithm)*. ...

$(1-\epsilon)\cdot OPT$.

Find out: does this work, since n will keep increasing if we add more items?

$$value^*(x_i) = \left\lfloor \frac{value(x_i)}{\Delta} \right\rfloor$$

To prove the running time, we have that

$$value^*(x_i) \le \left\lfloor \frac{\max_{1\le i\le n} value(x_i)}{\Delta} \right\rfloor + 1 = \left\lfloor \frac{\max_{1\le i\le n} value(x_i)}{\frac{\varepsilon}{n} LB} \right\rfloor + 1 = \left\lfloor \frac{n}{\varepsilon} \right\rfloor + 1$$

It follows that $value^*(X)$, the total new value, is at most $n \cdot \left(\left\lfloor \frac{n}{\varepsilon} \right\rfloor + 1\right) = O\left(\frac{n^2}{\varepsilon}\right)$; hence, by theorem 4.2, the running time is $O\left(\frac{n^3}{\varepsilon}\right)$, which proves the running time.

To prove the remainder of the theorem, once again let S_{OPT} denote an optimal subset; that is, a subset of weight at most W such that $value(S_{opt}) = OPT$.
Let S^* denote the subset returned by the algorithm. Since we did not change the weights of the items, the subset S^* has weight at most W. Hence, the computed solution S^* is feasible. It remains to show that $value(S^*) \ge (1-\varepsilon) OPT$.

Because S^* is optimal for the new values, we have $value^*(S^*) \ge value^*(S_{OPT})$. Moreover,

$$\frac{value(x_i)}{\Delta} - 1 \le value^*(x_i) \le \frac{value(x_i)}{\Delta}, \quad \text{where } \Delta = \frac{\varepsilon}{n} LB. \text{ Hence, we have}$$

$$\begin{aligned}
value(S^*) &= \Sigma_{x_i \in S^*} value(x_i) \\
&\ge \Sigma_{x_i \in S^*} \Delta \; value^*(x_i) \\
&= \Delta \, \Sigma_{x_i \in S^*} value^*(x_i) \\
&\ge \Delta \cdot \Sigma_{x_i \in S_{OPT}} value^*(x_i) \\
&\ge \Delta \cdot \left(\Sigma_{x_i \in S_{OPT}} \frac{value(x_i)}{\Delta} - 1 \right) \\
&= \left(\Sigma_{x_i \in S_{OPT}} value(x_i) \right) - |S_{OPT}| \Delta \\
&\ge \left(\Sigma_{x_i \in S_{OPT}} value(x_i) \right) - n \cdot \Delta \\
&\ge OPT - \varepsilon \cdot LB \\
&\ge OPT - \varepsilon \cdot OPT
\end{aligned}$$

$value(x_i) \ge \Delta \, value^*(x_i)$

$\boxed{value(x_i) \le (value^*(x_i) + 1)\Delta}$

Thus, $value(S^*) \ge (1-\varepsilon) \cdot OPT$.

Hence, theorem 4.3 still holds.