Exercise 4.3

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(i) Algorithm Weighted-Edge-Cover (G, w):

1. Solve the relaxed LP corresponding to the problem:

$$\text{minimize } \sum_{\{ij \mid e_{ij} \in E\}} x_{ij} \cdot w(e_{ij})$$

$$\text{subject to } \sum_{1 \le j \le |V|} x_{ij} \ge 1 \quad \text{forall } 1 \le i \le |V|$$

$$0 \le x_{ij} \le 1 \quad \text{forall } 1 \le i \le |V| \text{ and } 1 \le j \le |V|$$

We ignore that $w(e)$ is undefined for e_{ij} where there is no edge between vertices i and j. Alternatively, we could set the weight of such non-existing edges to infinity.

2. $C \leftarrow \{e_{ij} \in E \mid x_{ij} \ge \frac{1}{3}\}$

3. return C

This algorithm produces a correct cover, because, for all vertices, we have that (by the first constraint), the sum over the decision variables for edges incident to that vertex is at least 1. Hence, if there are at most three edges incident to every vertex, the average value of the decision variables of a vertex is at least $\frac{1}{3}$, which implies that at least one edge's decision variable has value $\frac{1}{3}$; since line 2 includes the corresponding edge in the cover, we have that, for every vertex, at least one edge incident to it is included in the cover. This proves the correctness of the cover.

To see why the required approximation ratio is achieved, we consider the following:

may perhaps better be read as $\{ij \mid e_{ij} \in E\}$

$$\begin{aligned}
\text{value}(C) &= \sum_{e_{ij} \in C} w(e_{ij}) \\
&\le \sum_{e_{ij} \in C} w(e_{ij})\, x_{ij} \cdot 3 \\
&= 3 \sum_{e_{ij} \in C} w(e_{ij})\, x_{ij} \\
&\le 3 \sum_{e_{ij} \in E} w(e_{ij})\, x_{ij} \\
&\le 3 \cdot OPT_{\text{relaxed}} \quad \leftarrow \text{the solution to the relaxed LP} \\
&\le 3 \cdot OPT
\end{aligned}$$

which proves the approximation ratio is 3.

(ii) We partition $(0, 5]$ into intervals of length Δ, where Δ is a parameter that we will pick later. Thus, we obtain a collection of intervals $(0, \Delta], (\Delta, 2\Delta], (2\Delta, 3\Delta]$, etc. We then replace each $w(e_{ij})$ by the value δ such that $w(e_{ij})$ lies in the δ-th interval: $((\delta-1)\Delta, \delta\Delta)$. In other words, we define $w'(e_{ij})$ as

$$w'(e_{ij}) = \left\lceil \frac{w(e_{ij})}{\Delta} \right\rceil$$

Note that, by rounding a value to the right endpoint of the interval it is contained in, we change the weight by less than Δ. We then scale the whole problem by dividing every weight by the same amount. We pick Δ based on the intuition that the error we induce in each value is at most Δ. To obtain a PTAS, the computed solution must have total weight at most $(1+\varepsilon)\,OPT$. In other words, the total error of the solution should be at most $\varepsilon \cdot OPT$. If the error in an individual value is at most Δ, then, since a (optimal) solution consists of at most $|V|$ (actually: $|V|-1$) edges, the total error of a solution is at most $\Delta \cdot |V|$. Thus, we want to choose Δ such that $\Delta |V| = \varepsilon\, OPT$, which suggests to pick $\Delta = \frac{\varepsilon\, OPT}{|V|}$.

However, we do not know OPT. To overcome this problem, we use a suitable lower bound LB instead of OPT. Clearly, we have that $OPT \ge \frac{|V|}{2} \cdot 0.5 = \frac{|V|}{4}$, since each edge can cover at most 2 vertices (and hence, we need at least $\frac{|V|}{2}$ edges to be picked) and the weight of each edge is at least 0.5. Note that, by working with LB instead of OPT, the interval size (and hence, the error) can only become smaller.

PTAS-WEC(G, ε)

	running time
1. $(V, E) \leftarrow G$	$O(1)$
2. let $LB \leftarrow \frac{\lvert V\rvert}{4}$	$O(\lvert V\rvert)$
3. let $\Delta \leftarrow \frac{\varepsilon \cdot LB}{\lvert V\rvert}$ $\left(= \frac{\varepsilon \cdot \lvert V\rvert}{\lvert V\rvert \cdot 4} = \frac{\varepsilon}{4}\right)$	$O(\lvert V\rvert)$
4. For all $e_{ij} \in E$, let $w'(e_{ij}) = \left\lceil \frac{w(e_{ij})}{\Delta} \right\rceil$	$O(\lvert E\rvert)$
5. Compute a subset $C \subseteq E$ of minimal weight with algorithm IntegerWeightEdgeCover (G), using the new values $w'(e_{ij})$ instead of $w(e_{ij})$.	$O\left(2^{w_{max}}(\lvert V\rvert+\lvert E\rvert)\right) = O\left(2^{\frac{5}{\Delta}}(\lvert V\rvert+\lvert E\rvert)\right) = O\left(2^{\frac{5\lvert V\rvert \cdot 4}{\varepsilon \lvert V\rvert}}(\lvert V\rvert+\lvert E\rvert)\right) = O\left(2^{\frac{20}{\varepsilon}}(\lvert V\rvert+\lvert E\rvert)\right)$
6. return C	$O(1)$

total running time: $O\left(2^{\frac{20}{\varepsilon}}(|V|+|E|)\right)$ → polynomial in $|V|+|E|$

proof of approximation ratio:

Let C_{opt} denote an optimal subset of edges, such that value$(C_{opt}) = OPT$. Let C' denote the subset returned by the algorithm. Such a subset will still cover all vertices, and hence, must be feasible. It remains to show that value$(C') \le (1+\varepsilon)\,OPT$

Because C' is optimal for the new weights, we have $w'(C') \le w'(C_{opt})$. Moreover,

$$\frac{w(e_{ij})}{\Delta} \le w'(e_{ij}) \le \frac{w(e_{ij})}{\Delta} + 1,$$

where $\Delta = \frac{\varepsilon \cdot LB}{|V|}$. Thus, we have

$$\begin{aligned}
w(C') &= \sum_{e_{ij} \in C'} w(e_{ij}) \\
&\le \sum_{e_{ij} \in C'} \Delta \cdot w'(e_{ij}) \\
&= \Delta \sum_{e_{ij} \in C'} w'(e_{ij}) \\
&\le \Delta \sum_{e_{ij} \in C_{opt}} w'(e_{ij}) \\
&\le \Delta \sum_{e_{ij} \in C_{opt}} \left(\frac{w(e_{ij})}{\Delta} + 1 \right) \\
&= \left(\sum_{e_{ij} \in C_{opt}} w(e_{ij}) \right) + \Delta \cdot \sum_{e_{ij} \in C_{opt}} 1 \\
&\le \left(\sum_{e_{ij} \in C_{opt}} w(e_{ij}) \right) + \Delta \cdot |V| \\
&= \left(\sum_{e_{ij} \in C_{opt}} w(e_{ij}) \right) + \frac{\varepsilon \cdot |V|}{4 \cdot |V|} |V| \\
&= OPT + \frac{\varepsilon}{4} |V| \\
&\le OPT + \varepsilon\, OPT,
\end{aligned}$$

which proves the approximation ratio is $1+\varepsilon$. □