Lectures 4.5 & 4.6

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To prove that the algorithm is a PTAS, we need to show that:

1. The output is valid: the selected subset S must have total weight $\leq W$;
2. $KnapsackPTAS(X, W, \epsilon) \geq (1-\epsilon) \cdot OPT$
3. The running time is polynomial in n

Since the weights have not been changed as part of the approximation scheme, the weight of the subset reported by the algorithm remains at most W; therefore, the first condition is satisfied.

Let S be the subset of items computed by the algorithm, which is optimal for $value^*$.
Let S_{opt} be the optimal subset of the original values.
We need to show that $value(S) \geq \cdots \geq (1-\epsilon) \cdot value(S_{opt}) = (1-\epsilon) \cdot OPT$.

We already know that $value^*(S) \geq value^*(S_{opt})$ and that $value^*(x_i) = \left\lceil \frac{value(x_i)}{\Delta} \right\rceil$, where $\Delta = \frac{\epsilon}{n} \cdot LB$ and $LB = \max_{x_i \in X} value(x_i) \leq OPT$. Now, we have that $\frac{value(x_i)}{\Delta} \leq value^*(x_i) \leq \frac{value(x_i)}{\Delta} + 1$.

We get

$$value(S) = \sum_{x_i \in S} value(x_i) \geq \sum_{x_i \in S} \Delta \cdot (value^*(x_i) - 1) = \Delta \cdot \left(\sum_{x_i \in S} value^*(x_i) \right) - |S| \cdot \Delta$$
$$\geq \Delta \cdot \left(\sum_{x_i \in S_{opt}} value^*(x_i) \right) - n \cdot \Delta \geq \left(\sum_{x_i \in S_{opt}} value(x_i) \right) - n \cdot \Delta = \mathrm{value}(\mathrm{S_{opt}}) - \mathrm{n} \cdot \Delta$$
$$= \mathrm{value}(\mathrm{S_{opt}}) - \epsilon \cdot \mathrm{LB} \geq \mathrm{value}(\mathrm{S_{opt}}) - \epsilon \cdot \mathrm{OPT} = (1-\epsilon) \cdot \mathrm{OPT}$$

This proves out algorithm has the right approximation ratio for a PTAS: $KnapsackPTAS(X, W, \epsilon) \geq (1-\epsilon) \cdot OPT$.

To show the third point, we consider that:

- The re-computation of item values takes $O(n)$ time.
- Returning the subset S of items computed in the exact algorithm but with the original values takes $O(n)$ time as well.
- To show the running time of the optimal algorithm, we consider that the running time of the optimal algorithm is $O(n\, V_{tot})$, where V_{tot} is the total value of the items in X^*. From the definition of Δ, we find that $V_{tot} = \sum_{x_i \in X} value^*(x_i) \leq n \cdot \max_{\mathrm{i}} value^*(x_i) = n \cdot \max_{\mathrm{i}} \left\lceil \frac{value^*(x_i)}{\Delta} \right\rceil = n \cdot \left\lceil \max_i \frac{value(x_i)}{\Delta} \right\rceil = n \cdot \left\lceil \max_i \frac{value(x_i)}{\frac{\epsilon}{\mathrm{n}} \cdot LB} \right\rceil = n \cdot \left\lceil \frac{n}{\epsilon} \right\rceil = O\left(\frac{n^2}{\epsilon}\right)$, which gives that the running time of the optimal algorithm is $O\left(\frac{n^3}{\epsilon}\right)$.

All in all, we can conclude that the running time is polynomial in n and in $\frac{1}{\epsilon}$. Hence, $KnapsackPTAS$ is an FPTAS.