Exercise 9.2

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we do not need to memorize the probability - equations by heart

To determine an appropriate value of k, we rewrite the proof of Lemma 9.6.

Only changes will be listed here:

we define X_i and Y_i as

$$X_i := \begin{cases} 1 & \text{if rank}(a_{r_i}) > \lceil 6(n+1)/10 \rceil \\ 0 & \text{otherwise} \end{cases}$$

$$Y_i := \begin{cases} 1 & \text{if rank}(a_{r_i}) < \lfloor 4(n+1)/10 \rfloor \\ 0 & \text{otherwise} \end{cases}$$

Now suppose the item we report is not a $\frac{1}{10}$-approximate median. Then either its rank is greater than $\lceil 6(n+1)/10 \rceil$ or its rank is smaller than $\lfloor 4(n+1)/10 \rfloor$. In the former case, at least half of the items in J must have rank greater than $\lceil 6(n+1)/10 \rceil$, and so $X \geq \frac{k}{2}$. Similarly, in the latter case, we have $Y \geq \frac{k}{2}$.

Observe that $E[X_i] = E[Y_i] = \frac{4}{10}$ because r_i is chosen uniformly at random. Hence,

$$E[X] = \sum_{i=1}^{k} E[X_i] = \frac{4k}{10} = \frac{2k}{5}$$

$\delta = \frac{1}{4}$ in Chernoff bound

Lemma 9.5 thus gives $Pr[X \geq \frac{k}{2}] = Pr[X \geq \frac{5}{4} E[X]] \leq \left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}} \right)^{\frac{2k}{5}}$

Similarly, $Pr[Y \geq \frac{k}{2}] \leq \left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}} \right)^{\frac{2k}{5}}$

Hence, $Pr[\text{algorithm reports a } \frac{1}{10}\text{-approximate median}] = 1 - 2 \cdot \left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}} \right)^{\frac{2k}{5}}$

We want $Pr[\text{algorithm reports a } \frac{1}{10}\text{-approximate median}] \geq 0.95$.

Thus, we need

$$1 - 2 \cdot \left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}} \right)^{\frac{2k}{5}} \geq 0.95$$

$$0.05 \geq 2 \left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}} \right)^{\frac{2k}{5}}$$

$$0.025 \geq \left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}} \right)^{\frac{2k}{5}}$$

$$\log_{\left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}\right)} (0.025) \leq \frac{2k}{5}$$

$$\frac{\log_2(0.025)}{\log_2\left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}\right)} \leq \frac{2k}{5}$$

$$\frac{5}{2} \frac{\log_2(0.025)}{\log_2\left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}\right)} \leq k$$

Thus, $k = \left\lceil \frac{5}{2} \frac{\log_2(0.025)}{\log_2\left(\frac{e^{\frac{1}{4}}}{\left(\frac{5}{4}\right)^{\frac{5}{4}}}\right)} \right\rceil$ is the minimal value of k for which the requested probability is reached.

$= 319$